7-5 Operations with Radical Expressions

Main Ideas

- Simplify radical expressions.
- Add, subtract, multiply, and divide radical expressions.

New Vocabulary

rationalizing the denominator

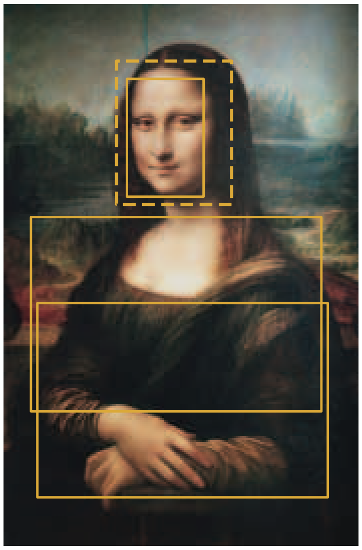
like radical expressions

conjugates

GET READY for the Lesson

Golden rectangles have been used by artists and architects to create beautiful designs. For example, if you draw a rectangle around the Mona Lisa's face, the resulting quadrilateral is the golden rectangle. The ratio of the lengths of the sides of a golden rectangle is $\frac{2}{\sqrt{5}-1}$.

In this lesson, you will learn how to simplify radical expressions like $\frac{2}{\sqrt{5}-1}$.



Simplify Radicals The properties you have used to simplify radical expressions involving square roots also hold true for expressions involving nth roots.

KEY CONCEPT — Properties of Radicals

For any real numbers a and b and any integer $n > 1$, the following properties hold true.

Property	Words	Examples
Product Property	**1.** If n is even and a and b are both nonnegative, then $\sqrt[n]{ab} = \sqrt[n]{a} \cdot \sqrt[n]{b}$, and **2.** If n is odd, then $\sqrt[n]{ab} = \sqrt[n]{a} \cdot \sqrt[n]{b}$.	$\sqrt{2} \cdot \sqrt{8} = \sqrt{16}$, or 4, and $\sqrt[3]{3} \cdot \sqrt[3]{9} = \sqrt[3]{27}$, or 3
Quotient Property	$\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}$, if all roots are defined and $b \neq 0$.	$\frac{\sqrt[3]{54}}{\sqrt[3]{2}} = \sqrt[3]{\frac{54}{2}} = \sqrt[3]{27}$, or 3

Follow these steps to simplify a square root.

Step 1 Factor the radicand into as many squares as possible.

Step 2 Use the Product Property to isolate the perfect squares.

Step 3 Simplify each radical.

Gianni Dagli Orti/CORBIS

You can use the properties of radicals to write expressions in simplified form.

CONCEPT SUMMARY *Simplifying Radical Expressions*

A radical expression is in simplified form when the following conditions are met.

- The index n is as small as possible.
- The radicand contains no factors (other than 1) that are nth powers of an integer or polynomial.
- The radicand contains no fractions.
- No radicals appear in a denominator.

To eliminate radicals from a denominator or fractions from a radicand, you can use a process called **rationalizing the denominator**. To rationalize a denominator, multiply the numerator and denominator by a quantity so that the radicand has an exact root.

Study Tip

Rationalizing the Denominator

You may want to think of rationalizing the denominator as making the denominator a rational number.

EXAMPLE 1 Simplify Expressions

Simplify.

a. $\sqrt{16p^8q^7}$

$$\sqrt{16p^8q^7} = \sqrt{4^2 \cdot (p^4)^2 \cdot (q^3)^2 \cdot q} \qquad \text{Factor into squares where possible.}$$

$$= \sqrt{4^2} \cdot \sqrt{(p^4)^2} \cdot \sqrt{(q^3)^2} \cdot \sqrt{q} \qquad \text{Product Property of Radicals}$$

$$= 4p^4|q^3|\sqrt{q} \qquad \text{Simplify.}$$

However, for $\sqrt{16p^8q^7}$ to be defined, $16p^8q^7$ must be nonnegative. If that is true, q must be nonnegative, since it is raised to an odd power. Thus, the absolute value is unnecessary, and $\sqrt{16p^8q^7} = 4p^4q^3\sqrt{q}$.

b. $\sqrt{\dfrac{x^4}{y^5}}$

$$\sqrt{\frac{x^4}{y^5}} = \frac{\sqrt{x^4}}{\sqrt{y^5}} \qquad \text{Quotient Property}$$

$$= \frac{\sqrt{(x^2)^2}}{\sqrt{(y^2)^2 \cdot y}} \qquad \text{Factor into squares.}$$

$$= \frac{\sqrt{(x^2)^2}}{\sqrt{(y^2)^2} \cdot \sqrt{y}} \qquad \text{Product Property}$$

$$= \frac{x^2}{y^2\sqrt{y}} \qquad \sqrt{(x^2)^2} = x^2$$

$$= \frac{x^2}{y^2\sqrt{y}} \cdot \frac{\sqrt{y}}{\sqrt{y}} \qquad \text{Rationalize the denominator.}$$

$$= \frac{x^2\sqrt{y}}{y^3} \qquad \sqrt{y} \cdot \sqrt{y} = y$$

c. $\sqrt[5]{\dfrac{5}{4a}}$

$$\sqrt[5]{\frac{5}{4a}} = \frac{\sqrt[5]{5}}{\sqrt[5]{4a}} \qquad \text{Quotient Property}$$

$$= \frac{\sqrt[5]{5}}{\sqrt[5]{4a}} \cdot \frac{\sqrt[5]{8a^4}}{\sqrt[5]{8a^4}} \qquad \text{Rationalize the denominator.}$$

$$= \frac{\sqrt[5]{5 \cdot 8a^4}}{\sqrt[5]{4a \cdot 8a^4}} \qquad \text{Product Property}$$

$$= \frac{\sqrt[5]{40a^4}}{\sqrt[5]{32a^5}} \qquad \text{Multiply.}$$

$$= \frac{\sqrt[5]{40a^4}}{2a} \qquad \sqrt[5]{32a^5} = 2a$$

CHECK Your Progress Simplify.

1A. $\sqrt{36r^5s^{10}}$ **1B.** $\sqrt{\frac{m^9}{n^7}}$ **1C.** $\sqrt[4]{\frac{3x}{2}}$

Operations with Radicals You can use the Product and Quotient Properties to multiply and divide some radicals, respectively.

EXAMPLE Multiply Radicals

2 **Simplify $6\sqrt[3]{9n^2} \cdot 3\sqrt[3]{24n}$.**

$$
\begin{aligned}
6\sqrt[3]{9n^2} \cdot 3\sqrt[3]{24n} &= 6 \cdot 3 \cdot \sqrt[3]{9n^2 \cdot 24n} && \text{Product Property of Radicals} \\
&= 18 \cdot \sqrt[3]{2^3 \cdot 3^3 \cdot n^3} && \text{Factor into cubes where possible.} \\
&= 18 \cdot \sqrt[3]{2^3} \cdot \sqrt[3]{3^3} \cdot \sqrt[3]{n^3} && \text{Product Property of Radicals} \\
&= 18 \cdot 2 \cdot 3 \cdot n \text{ or } 108n && \text{Multiply.}
\end{aligned}
$$

CHECK Your Progress

Simplify.

2A. $5\sqrt[4]{24x^3} \cdot 4\sqrt[4]{54x}$ **2B.** $7\sqrt[3]{75a^4} \cdot 3\sqrt[3]{45a^2}$

Can you add radicals in the same way that you multiply them? In other words, if $\sqrt{a} \cdot \sqrt{a} = \sqrt{a \cdot a}$, does $\sqrt{a} + \sqrt{a} = \sqrt{a + a}$?

ALGEBRA LAB

Adding Radicals

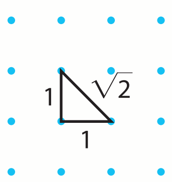
You can use dot paper to show the sum of two like radicals, such as $\sqrt{2} + \sqrt{2}$.

Step 1 First, find a segment of length $\sqrt{2}$ units by using the Pythagorean Theorem with the dot paper.

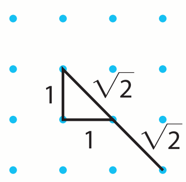
$a^2 + b^2 = c^2$

$1^2 + 1^2 = c^2$

$2 = c^2$



Step 2 Extend the segment to twice its length to represent $\sqrt{2} + \sqrt{2}$.



ANALYZE THE RESULTS

1. Is $\sqrt{2} + \sqrt{2} = \sqrt{2 + 2}$ or 2? Justify your answer using the geometric models above.

2. Use this method to model other irrational numbers. Do these models support your conjecture?

Study Tip

Simplifying Radicals

In general, $\sqrt{a} + \sqrt{b} \neq \sqrt{a + b}$. In fact, $\sqrt{a} + \sqrt{b} = \sqrt{a + b}$ only when $a = 0$, $b = 0$, or both $a = 0$ and $b = 0$.

You add radicals in the same manner as adding monomials. That is, you can add only the like terms or like radicals. Two radical expressions are called **like radical expressions** if both the indices and the radicands are alike.

Like: $2\sqrt[4]{3a}$ and $5\sqrt[4]{3a}$ Radicands are $3a$; indices are 4.

Unlike: $\sqrt{3}$ and $\sqrt[3]{3}$ Different indices

$\sqrt[4]{5x}$ and $\sqrt[4]{5}$ Different radicands

Concepts in Motion
Animation
algebra2.com

EXAMPLE Add and Subtract Radicals

3 **Simplify $2\sqrt{12} - 3\sqrt{27} + 2\sqrt{48}$.**

$2\sqrt{12} - 3\sqrt{27} + 2\sqrt{48}$

$= 2\sqrt{2^2 \cdot 3} - 3\sqrt{3^2 \cdot 3} + 2\sqrt{2^2 \cdot 2^2 \cdot 3}$ Factor using squares.

$= 2\sqrt{2^2} \cdot \sqrt{3} - 3\sqrt{3^2} \cdot \sqrt{3} + 2\sqrt{2^2} \cdot \sqrt{2^2} \cdot \sqrt{3}$ Product Property

$= 2 \cdot 2 \cdot \sqrt{3} - 3 \cdot 3 \cdot \sqrt{3} + 2 \cdot 2 \cdot 2 \cdot \sqrt{3}$ $\sqrt{2^2} = 2$, $\sqrt{3^2} = 3$

$= 4\sqrt{3} - 9\sqrt{3} + 8\sqrt{3}$ Multiply.

$= 3\sqrt{3}$ Combine like radicals.

CHECK Your Progress Simplify.

3A. $3\sqrt{8} + 5\sqrt{32} - 4\sqrt{18}$ **3B.** $5\sqrt{12} - 2\sqrt{27} + 6\sqrt{108}$

Just as you can add and subtract radicals like monomials, you can multiply radicals using the FOIL method as you do when multiplying binomials.

EXAMPLE Multiply Radicals

4 **Simplify.**

a. $(3\sqrt{5} - 2\sqrt{3})(2 + \sqrt{3})$

$$(3\sqrt{5} - 2\sqrt{3})(2 + \sqrt{3}) = \overset{F}{3\sqrt{5} \cdot 2} + \overset{O}{3\sqrt{5} \cdot \sqrt{3}} - \overset{I}{2\sqrt{3} \cdot 2} - \overset{L}{2\sqrt{3} \cdot \sqrt{3}}$$

$= 6\sqrt{5} + 3\sqrt{5 \cdot 3} - 4\sqrt{3} - 2\sqrt{3^2}$ Product Property

$= 6\sqrt{5} + 3\sqrt{15} - 4\sqrt{3} - 6$ $2\sqrt{3^2} = 2 \cdot 3$ or 6

b. $(5\sqrt{3} - 6)(5\sqrt{3} + 6)$

$(5\sqrt{3} - 6)(5\sqrt{3} + 6) = 5\sqrt{3} \cdot 5\sqrt{3} + 5\sqrt{3} \cdot 6 - 6 \cdot 5\sqrt{3} - 6 \cdot 6$ FOIL

$= 25\sqrt{3^2} + 30\sqrt{3} - 30\sqrt{3} - 36$ Multiply.

$= 75 - 36$ $25\sqrt{3^2} = 25 \cdot 3$ or 75

$= 39$ Subtract.

CHECK Your Progress

4A. $(4\sqrt{2} + 2\sqrt{6})(\sqrt{5} - 3)$ **4B.** $(3\sqrt{5} + 4)(3\sqrt{5} - 4)$

Study Tip

Conjugates

The product of conjugates of the form $a\sqrt{b} + c\sqrt{d}$ and $a\sqrt{b} - c\sqrt{d}$ is always a rational number.

Binomials like those in Example 4b, of the form $a\sqrt{b} + c\sqrt{d}$ and $a\sqrt{b} - c\sqrt{d}$, where a, b, c, and d are rational numbers, are called **conjugates** of each other. You can use conjugates to rationalize denominators.

EXAMPLE 5 Use a Conjugate to Rationalize a Denominator

Simplify $\dfrac{1-\sqrt{3}}{5+\sqrt{3}}$.

$$\frac{1-\sqrt{3}}{5+\sqrt{3}} = \frac{(1-\sqrt{3})(5-\sqrt{3})}{(5+\sqrt{3})(5-\sqrt{3})}$$
Multiply by $\frac{5-\sqrt{3}}{5-\sqrt{3}}$ because $5-\sqrt{3}$ is the conjugate of $5+\sqrt{3}$.

$$= \frac{1\cdot 5 - 1\cdot\sqrt{3} - \sqrt{3}\cdot 5 + (\sqrt{3})^2}{5^2 - (\sqrt{3})^2}$$
FOIL

Difference of squares

$$= \frac{5-\sqrt{3}-5\sqrt{3}+3}{25-3}$$
Multiply.

$$= \frac{8-6\sqrt{3}}{22}$$
Combine like terms.

$$= \frac{4-3\sqrt{3}}{11}$$
Divide numerator and denominator by 2.

CHECK Your Progress **Simplify.**

5A. $\dfrac{4+\sqrt{2}}{5-\sqrt{2}}$ **5B.** $\dfrac{3-2\sqrt{5}}{6+\sqrt{5}}$

Online Personal Tutor at algebra2.com

CHECK Your Understanding

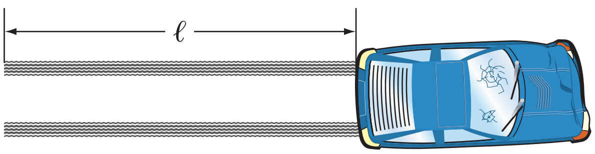
Example 1 (pp. 409–410)

Simplify.

1. $5\sqrt{63}$ **2.** $\sqrt[4]{16x^5y^4}$ **3.** $\sqrt{75x^3y^6}$

4. $\sqrt{\dfrac{7}{8y}}$ **5.** $\sqrt{\dfrac{a^7}{b^9}}$ **6.** $\sqrt[3]{\dfrac{2}{3x}}$

LAW ENFORCEMENT **For Exercises 7 and 8, use the following information.**
Under certain conditions, a police accident investigator can use the formula $s = \dfrac{10\sqrt{\ell}}{\sqrt{5}}$ to estimate the speed s of a car in miles per hour based on the length ℓ in feet of the skid marks it left.



7. Write the formula without a radical in the denominator.

8. How fast was a car traveling that left skid marks 120 feet long?

Examples 2–5 (pp. 410–412)

Simplify.

9. $(-2\sqrt{15})(4\sqrt{21})$ **10.** $\sqrt{2ab^2}\cdot\sqrt{6a^3b^2}$ **11.** $\dfrac{\sqrt[3]{625}}{\sqrt[3]{25}}$

12. $\sqrt{3} - 2\sqrt[4]{3} + 4\sqrt{3} + 5\sqrt[4]{3}$ **13.** $3\sqrt[3]{128} + 5\sqrt[3]{16}$

14. $(3-\sqrt{5})(1+\sqrt{3})$ **15.** $(2+\sqrt{2})(2-\sqrt{2})$

16. $\dfrac{1+\sqrt{5}}{3-\sqrt{5}}$ **17.** $\dfrac{4-\sqrt{7}}{3+\sqrt{7}}$

Exercises

HOMEWORK HELP

For Exercises	See Examples
18–23	1
34–35	2
36–41	3
42–45	4
46–51	5

Simplify.

18. $\sqrt{243}$ **19.** $\sqrt{72}$ **20.** $\sqrt[3]{54}$ **21.** $\sqrt[4]{96}$

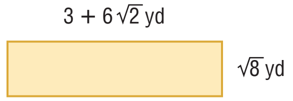
22. $\sqrt{50x^4}$ **23.** $\sqrt[3]{16y^3}$ **24.** $\sqrt{18x^2y^3}$ **25.** $\sqrt{40a^3b^4}$

26. $3\sqrt[3]{56y^6z^3}$ **27.** $2\sqrt[3]{24m^4n^5}$ **28.** $\sqrt[4]{\frac{1}{81}c^5d^4}$ **29.** $\sqrt[5]{\frac{1}{32}w^6z^7}$

30. $\sqrt[3]{\frac{3}{4}}$ **31.** $\sqrt[4]{\frac{2}{3}}$ **32.** $\sqrt{\frac{a^4}{b^3}}$ **33.** $\sqrt{\frac{4r^8}{t^9}}$

34. $(3\sqrt{12})(2\sqrt{21})$ **35.** $(-3\sqrt{24})(5\sqrt{20})$

36. GEOMETRY Find the perimeter and area of the rectangle.



37. GEOMETRY Find the perimeter of a regular pentagon whose sides measure $(2\sqrt{3} + 3\sqrt{12})$ feet.

Simplify.

38. $\sqrt{12} + \sqrt{48} - \sqrt{27}$ **39.** $\sqrt{98} - \sqrt{72} + \sqrt{32}$

40. $\sqrt{3} + \sqrt{72} - \sqrt{128} + \sqrt{108}$ **41.** $5\sqrt{20} + \sqrt{24} - \sqrt{180} + 7\sqrt{54}$

42. $(5 + \sqrt{6})(5 - \sqrt{2})$ **43.** $(3 + \sqrt{7})(2 + \sqrt{6})$

44. $(\sqrt{11} - \sqrt{2})^2$ **45.** $(\sqrt{3} - \sqrt{5})^2$

46. $\frac{7}{4 - \sqrt{3}}$ **47.** $\frac{\sqrt{6}}{5 + \sqrt{3}}$ **48.** $\frac{-2 - \sqrt{3}}{1 + \sqrt{3}}$

49. $\frac{2 + \sqrt{2}}{5 - \sqrt{2}}$ **50.** $\frac{x + 1}{\sqrt{x^2 - 1}}$ **51.** $\frac{x - 1}{\sqrt{x} - 1}$

52. What is $\sqrt{39}$ divided by $\sqrt{26}$?

53. Divide $\sqrt{14}$ by $\sqrt{35}$.

AMUSEMENT PARKS For Exercises 54 and 55, use the following information.
The velocity v in feet per second of a roller coaster at the bottom of a hill is related to the vertical drop h in feet and the velocity v_0 in feet per second of the coaster at the top of the hill by the formula $v_0 = \sqrt{v^2 - 64h}$.

54. Explain why $v_0 = v - 8\sqrt{h}$ is not equivalent to the given formula.

55. What velocity must a coaster have at the top of a 225-foot hill to achieve a velocity of 120 feet per second at the bottom?

SPORTS For Exercises 56 and 57, use the following information.
A ball that is hit or thrown horizontally with a velocity of v meters per second will travel a distance of d meters before hitting the ground, where $d = v\sqrt{\frac{h}{4.9}}$ and h is the height in meters from which the ball is hit or thrown.

56. Use the properties of radicals to rewrite the formula.

57. How far will a ball that is hit with a velocity of 45 meters per second at a height of 0.8 meter above the ground travel before hitting the ground?

EXTRA PRACTICE
See pages 906 and 932.

Math Online
Self-Check Quiz at
algebra2.com

H.O.T. Problems

58. REASONING Determine whether the statement $\frac{1}{\sqrt[n]{a}} = \sqrt[n]{a}$ is *sometimes*, *always*, or *never* true. Explain.

59. OPEN ENDED Write a sum of three radicals that contains two like terms. Explain how you would combine the terms. Defend your answer.

Six Flags

60. **FIND THE ERROR** Ethan and Alexis are simplifying $\frac{4+\sqrt{5}}{2-\sqrt{5}}$. Who is correct? Explain your reasoning.

Ethan

$$\frac{4+\sqrt{5}}{2-\sqrt{5}} = \frac{4+\sqrt{5}}{2-\sqrt{5}} \cdot \frac{2+\sqrt{5}}{2+\sqrt{5}}$$
$$= \frac{13+6\sqrt{5}}{-1}$$

Alexis

$$\frac{4+\sqrt{5}}{2-\sqrt{5}} = \frac{4+\sqrt{5}}{2-\sqrt{5}} \cdot \frac{4-\sqrt{5}}{4-\sqrt{5}}$$
$$= \frac{11}{13-6\sqrt{5}}$$

61. ***Writing in Math*** Refer to the information given on page 408 to explain how radical expressions relate to the Mona Lisa. Use the properties in this lesson to explain how you could rewrite the radical expression.

STANDARDIZED TEST PRACTICE

62. **ACT/SAT** The expression $\sqrt{180a^2b^8}$ is equivalent to which of the following?

A $5\sqrt{6}|a|b^4$

B $6\sqrt{5}|a|b^4$

C $3\sqrt{10}|a|b^4$

D $36\sqrt{5}|a|b^4$

63. **REVIEW** When the number of a year is divisible by 4, then a leap year occurs. However, when the year is divisible by 100, then a leap year does not occur unless the year is divisible by 400. Which is *not* an example of a leap year?

F 1884

G 1900

H 1904

J 1940

Spiral Review

Simplify. (Lesson 7-4)

64. $\sqrt{144z^8}$

65. $\sqrt[3]{216a^3b^9}$

66. $\sqrt{(y+2)^2}$

67. Graph $y \le \sqrt{x+1}$. (Lesson 7-3)

68. **ELECTRONICS** There are three basic things to be considered in an electrical circuit: the flow of the electrical current I, the resistance to the flow Z, called impedance, and electromotive force E, called voltage. These quantities are related in the formula $E = I \cdot Z$. The current of a circuit is to be $35 - 40j$ amperes. Electrical engineers use the letter j to represent the imaginary unit. Find the impedance of the circuit if the voltage is to be $430 - 330j$ volts. (Lesson 5-4)

Find the inverse of each matrix, if it exists. (Lesson 4-7)

69. $\begin{bmatrix} 8 & 6 \\ 7 & 5 \end{bmatrix}$

70. $\begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}$

71. $\begin{bmatrix} 8 & 4 \\ 6 & 3 \end{bmatrix}$

GET READY for the Next Lesson

PREREQUISITE SKILL Evaluate each expression.

72. $2\left(\frac{1}{8}\right)$

73. $3\left(\frac{1}{6}\right)$

74. $\frac{1}{2} + \frac{1}{3}$

75. $\frac{1}{3} + \frac{3}{4}$

76. $\frac{1}{8} + \frac{5}{12}$

77. $\frac{5}{6} - \frac{1}{5}$

78. $\frac{5}{8} - \frac{1}{4}$

79. $\frac{1}{4} - \frac{2}{3}$